International Journal of Theoretical Physics, Vol. 22, No. 4, 1983

Decoupling Theorem of Quantum Field Theory in Minkowski Space[1]

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Received June 15, 1981

The decoupling theorem of quantum field theory with massive particles is proved in Minkowski space when all the masses of the theory are led to go to infinity. The theorem establishes the vanishing property, in the distributional sense, of (absolutely convergent) Feynman amplitudes in a model independent way with subtractions performed at the origin. This extends previous efforts dealing with the proof of the theorem in the Euclidean region.

1. INTRODUCTION

Numerous applications of the decoupling theorem have been and are being carried out in the literature (Appelquist and Carazzonne, 1975; Poggio et al., 1977; Collins et al., 1978; Toussaint, 1978; Kazama and Yao, 1980a,b; Hagiwara and Nakazawa, 1981; Ovrut and Schnitzer, 1981). This important theorem states that (renormalized) Feynman amplitudes involving "heavy" masses may be neglected. This has interesting consequences on modern and future (Huff and Prewett, 1979) field theory models involving heavy masses much higher than available energies in experiments. Rigorous proofs of the decoupling theorem are available in the literature (Manoukian, 1981; Ambjørn, 1979) which deals with the Euclidean region. We extend this theorem, for quantum field theory with massive particles, to Minkowski space for the cases where all the masses of the theory are led to go to infinity with subtractions of renormalization carried out at the origin. We establish the vanishing property of (absolutely convergent) Feynman amplitudes in the distributional sense in a model independent way.

[1]Work supported by the Department of National Defence Award under CRAD No. 3610-637.

0020-7748/83/0400-0315$03.00/0

2. PROOF OF THE THEOREM

A Feynman amplitude, associated with some proper and connected graph G, in Minkowski space, may be written in the form

$$F_\varepsilon(P,\mu)=\int_{\mathbb{R}^{4n}} dK\, A(P,K,\mu,\varepsilon)\prod_{l=1}^{L} D_l^{-1}, \qquad \varepsilon>0 \tag{1}$$

$$D_l=\left[Q_l^2+\mu_l^2-i\varepsilon\left(\mathbf{Q}_l^2+\mu_l^2\right)\right] \tag{2}$$

$$K=\left(k_1^0,\ldots,k_n^3\right), \qquad P=\left(p_1^0,\ldots,p_m^3\right) \tag{3}$$

$$\mu=\left(\mu^1,\ldots,\mu^\rho\right) \tag{4}$$

where K is the set of the integration variables, P is the set of components of the independent external momenta, and μ is the set of masses in the theory. A is a polynomial in the elements in P, K, μ, in ε and may be even a polynomial in the $(\mu^i)^{-1}$ as well. The latter is well known to arise for theories with high spin fields. The Q_l are of the form $Q_l=\Sigma_j a_{lj}k_j+\Sigma_j b_{lj}p_j$, where the matrix $[a_{lj}]$ is assumed to be of rank n. In (2) we have adopted the $i\varepsilon$ prescription first introduced in Zimmermann (1968). Each μ_l (>0) in (2) coincides with one of the elements in (4). In our metric $Q^2=\mathbf{Q}^2-(Q^0)^2$. For $\varepsilon>0$, the integral in (1) is assumed to be absolutely convergent (a.c.). We prove the following theorem.

Theorem. For any $f(P)\in S(\mathbb{R}^{4m})$, with the latter denoting the Schwartz space (Schwartz, 1978), then with

$$T_\varepsilon(f;\eta)=\int_{\mathbb{R}^{4m}} dP\, f(P)F_\varepsilon(P,\eta\mu) \tag{5}$$

we have as a tempered distribution

$$\lim_{\eta\to\infty}\left(\lim_{\varepsilon\to 0} T_\varepsilon(f;\eta)\right)=0 \tag{6}$$

We introduce Feynman parameters to write $F_\varepsilon(P,\mu)$ as

$$F_\varepsilon(P,\mu)=(L-1)!\int_{\mathbb{R}^{4n}} dK\, A(P,K,\mu,\varepsilon)\int_{\mathsf{D}} d\alpha\left(\sum_{l=1}^{L}\alpha_l D_l\right)^{-L} \tag{7}$$

where $\mathsf{D}=\{\alpha=(\alpha_1,\ldots,\alpha_L)\colon \alpha_l\geqslant 0,\ \Sigma_{l=1}^{L}\alpha_l=1\}$. For $\varepsilon>0$ it is easily established (Zimmermann, 1968) that (7) is a.c. and hence by the Fubini–Tonelli theorem we may interchange the orders of integration over dK and $d\alpha$ and obtain, up to an overall multiplicative constant dependent on ε which has a well-defined limit for $\varepsilon \to 0$,

$$F_\varepsilon(P, \mu) = \int_{\mathrm{D}} d\alpha\, N(\alpha, P, \mu, \varepsilon) G_\varepsilon(\alpha, P)^{-t} \tag{8}$$

where $N(\alpha, P, \mu, \varepsilon)$ is rational in α and is a polynomial in ε and in the elements of P, μ and may be even a polynomial in the $(\mu^i)^{-1}$. t is some positive integer and

$$G_\varepsilon(\alpha, P) = pUp + M^2 - i\varepsilon(\mathbf{p}\cdot U\mathbf{p} + M^2)$$

$$M^2 = \sum_{l=1}^{L} \alpha_l \mu_l^2 \tag{9}$$

where U is rational in α (Zimmermann, 1968; Lowenstein and Speer, 1976). The following lemma is proved in the same manner as in Zimmermann (1968) and Lowenstein and Speer (1976) with some elementary modifications.

Lemma. (i) $\int_{\mathrm{D}} d\alpha\, N(\alpha, P, \mu, \varepsilon)$, (ii) $\int_{\mathbf{R}^{4n}} dP\, f(P) \int_{\mathrm{D}} d\alpha\, N(\alpha, P, \mu, \varepsilon) G_\varepsilon(\alpha, P)^{-t}$, $\varepsilon > 0$, (iii) $\int_{\mathrm{D}} d\alpha\, N_{abcd}(\alpha)$, are all a.c., where in a compact and standard notation,

$$N(\alpha, P, \mu, \varepsilon) = \sum_{a,b,c,d} \mu^a \mu^{-b} p^c \varepsilon^d N_{abcd}(\alpha) \tag{10}$$

and, in particular, $a = (a_1, \ldots, a_p)$, $c = (c_{01}, \ldots, c_{3m})$, $\mu^{-b} = (\mu^1)^{-b_1}, \ldots, (\mu^p)^{-b_p}$, with the $a_i, c_{\mu j}, b_j, d$ being finite nonnegative integers. Statement (iii) follows from (i) by the application of the so-called generalized Lagrange's interpolating (Zimmermann, 1968) formula.

We rewrite $F_\varepsilon(P, \mu)$ in (1) as

$$\begin{aligned} F_\varepsilon(P, \eta\mu) &= \int_{\mathbb{R}^{4n}} dK\, A(P, K, \eta\mu, \varepsilon) \prod_{l=1}^{L} D_l(P, K, \eta\mu, \varepsilon)^{-1} \\ &= (\eta)^{d_0} \int_{\mathbb{R}^{4n}} dK\, A\left(\frac{P}{\eta}, k, \mu, \varepsilon\right) \prod_{l=1}^{L} D_l\left(\frac{P}{\eta}, K, \mu, \varepsilon\right)^{-1} \\ &= (\eta)^{d_0} F_\varepsilon\left(\frac{P}{\eta}, \mu\right) \end{aligned} \tag{11}$$

where $d_0 \leqslant d(G)$. The exponent d_0 in (11) may be, in general, reduced over $d(G)$ if A is a polynomial in the $(\mu^i)^{-1}$ as well. Accordingly from (11) and (8), we may rewrite (5) as

$$T_\varepsilon(f;\eta) = (\eta)^{d_0} \int_{\mathbb{R}^{4m}} dP f(P) \int_{\mathrm{D}} d\alpha\, N\left(\alpha, \frac{P}{\eta}, \mu, \varepsilon\right) G_\varepsilon\left(\alpha, \frac{P}{\eta}\right)^{-t} \tag{12}$$

Following Lowenstein and Speer (1976) (see also Hepp, 1966), but with some modifications, we introduce a C^∞ function $\chi(x)$: $0 \leqslant \chi(x) \leqslant 1$ with $\chi(x) = 1$ for $-1/3 \leqslant x$ and $\chi(x) = 0$ for $x < -2/3$, with $x = (pUp/\eta^2\mu^2)$, $\mu = \min_j \mu_j$, and rewrite (12) as

$$T_\varepsilon(f;\eta) = T_\varepsilon^1(f;\eta) + T_\varepsilon^2(f;\eta) \tag{13}$$

$$T_\varepsilon^1(f;\eta) = (\eta)^{d_0} \int_{\mathbb{R}^{4m}} dP \int_{\mathrm{D}} d\alpha\, f(P) \chi\left(\frac{pUp}{\eta^2\mu^2}\right) N\left(\alpha, \frac{P}{\eta}, \mu, \varepsilon\right) \left[G_\varepsilon\left(\alpha, \frac{P}{\eta}\right)\right]^{-t} \tag{14}$$

$$T_\varepsilon^2(f;\eta) = (\eta)^{d_0+4m} \int_{\mathbb{R}^{4m}} dP \int_{\mathrm{D}} d\alpha\, f(\eta P) \left[1 - \chi\left(\frac{pUp}{\mu^2}\right)\right]$$
$$\times N(\alpha, P, \mu, \varepsilon) G_\varepsilon(\alpha, P)^{-t} \tag{15}$$

We note from (10) that (14) may be rewritten as

$$T_\varepsilon^1(f;\eta) = \sum_{a,b,c,d} \mu^{a-b} \eta^{d_0-|c|} \varepsilon^d \int_{\mathbb{R}^{4m}} dP \int_{\mathrm{D}} d\alpha\, p^c f(P) \chi\left(\frac{pUp}{\eta^2\mu^2}\right)$$
$$\times N_{abcd}(\alpha) \left[G_\varepsilon\left(\alpha, \frac{P}{\eta}\right)\right]^{-t} \tag{16}$$

where $|c| = |c_{01}| + \cdots |c_{3m}|$. For the integral (16) we have that [see definition of $\chi(x)$]:

$$\frac{pUp}{\eta^2} + M^2 \geqslant -\frac{2}{3}\mu^2 + \mu^2 \geqslant \frac{1}{3}\mu^2 \tag{17}$$

and

$$\left|G_\varepsilon\left(\alpha, \frac{P}{\eta}\right)\right|^{-1} \leqslant \left|G_0\left(\alpha, \frac{P}{\eta}\right)\right|^{-1} \leqslant 3/\mu^2 \tag{18}$$

Therefore

$$|T_\varepsilon^1(f;\eta)| \leqslant C \sum_{a,b,c,d} \mu^{a-b}\eta^{d_0-|c|}\varepsilon^d \int_{\mathbb{R}^{4m}} dP\,|p^c f(P)| \times \int_D d\alpha |N_{abcd}(\alpha)| < \infty \tag{19}$$

where $C>0$ and we have used part (iii) of the lemma, the fact that $|\chi(x)| \leqslant 1$ and the fact that $p^c f(P) \in S(\mathbb{R}^{4m})$. We note that $d \geqslant 0$. Also if $d(G)<0$, then $|c| \geqslant 0$; on the other hand, if $d(G) \geqslant 0$, then $|c| \geqslant d(G)+1$, with subtractions carried out at the origin, i.e., $d_0 - |c| < 0$. Accordingly from the Lebesgue dominated convergence theorem [note also (18)] we have

$$\lim_{\eta\to\infty} \lim_{\varepsilon\to 0} T_\varepsilon^1(f;\eta) = 0 \tag{20}$$

(Actually the limits may be even reversed.)

For the second integral $T_\varepsilon^2(f;\eta)$ we use a variation of an identity in Lowenstein and Speer (1976):

$$G_\varepsilon^{-t} = -\frac{(-1/2)^t}{(t-1)!}\left[(pUp)_\varepsilon^{-1} \sum_{j=1}^{m} p_j^\mu \frac{\partial}{\partial p_j^\mu}\right]^t \ln G_\varepsilon \tag{21}$$

where $(pUp)_\varepsilon = pUp - i\varepsilon \mathbf{p}\cdot U\mathbf{p}$. Substituting (21) in (15) using (10), integrating by parts, and using the vanishing property of $p^c f$, together with all its derivatives, at infinity we obtain for $T_\varepsilon^2(f;\eta)$

$$T_\varepsilon^2(f;\eta) = -\frac{(1/2)^t}{(t-1)!}(\eta)^{d_0+4m} \sum_{a,b,c,d} \mu^{a-b}\varepsilon^d \int_{\mathbb{R}^{4m}} dP \int_D d\alpha\, N_{abcd}(\alpha) \times \ln[G_\varepsilon(\alpha,P)]\left\{\left[\sum_{j=1}^{m} \frac{\partial}{\partial p_j^\mu}\frac{p_j^\mu}{(pUp)_\varepsilon}\right]^t p^c f(\eta P)\left[1-\chi\left(\frac{pUp}{\mu^2}\right)\right]\right\} \tag{22}$$

The expression in $\{\cdot\}$ may be explicitly (Lowenstein and Speer, 1976) written as a finite sum of the form

$$[(pUp)_\varepsilon]^{-t} \sum \eta^{c_i} h^i(\eta P)\chi_i(\alpha,P) \tag{23}$$

where the $c_i \geq 0$, $h^i \in S(\mathbb{R}^{4m})$, and the χ_i are bounded and vanish for (α, P) outside the set $\{(\alpha, P)\colon pUp/\mu^2 \leq -1/3\}$. Accordingly for the integral (22) we have

$$|(pUp)_\varepsilon| \geq |pUp| \geq 1/3\mu^2 \tag{24}$$

and hence we may bound (22) as

$$\begin{aligned} |T_\varepsilon^2(f;\eta)| \leq C' \sum_{a,b,c,d} \mu^{a-b}\varepsilon^d \sum_i \int_{\mathbb{R}^{4m}} dP\, \eta^{g_i} |h^i(\eta P)| \int_D d\alpha |N_{abcd}(\alpha)| \\ \times |\ln G_\varepsilon(\alpha, P)| \end{aligned} \tag{25}$$

with $C' > 0$ and finite g_i. Since $h^i \in S(\mathbb{R}^{4m})$, we may find positive integers N_1 and N_2 and a constant $H > 0$ such that

$$\begin{aligned} |h^i(\eta P)| &\leq H\left(\frac{\eta^2|p|^2}{\mu^2}+1\right)^{-N_1}\left(\frac{\eta|p|^2}{\mu^2}+1\right)^{-N_2} \\ &\leq H[a_0\eta^2+1]^{-N_1}\left(\frac{|p|^2}{\mu^2}+1\right)^{-N_2}, \qquad \eta > 1 \end{aligned} \tag{26}$$

where $|p|^2 = \sum_{i,\mu} |p_i^\mu|^2$ and where from (24) and the continuity condition of U on α (Lowenstein and Speer, 1976) we note that $a_0 > 0$. Accordingly by choosing N_1 *sufficiently large* we can make $\eta^{g_i}\eta^{-2N_1} \to 0$ for $\eta \to \infty$. On the other hand, with the $h^i(\eta p)$ replaced by $(|p|^2/\mu^2+1)^{-N_2}$, with N_2 chosen sufficiently large, we may apply the so-called Hironaka–Atiyah–Bernstein-Gel'fand theorem (Hironaka, 1964; Atiyah, 1970; Bernstein and Gel'fand, 1969) tailored to the problem at hand (Lowenstein and Speer, 1976) to establish that since the integral on the right-hand side of (25) without the $\ln G_\varepsilon$ is a.c. then the integral with this factor is also a.c. for all $\varepsilon \geq 0$, and hence by using in the process the Lebesgue dominated convergence theorem we obtain

$$\lim_{\eta\to\infty} \lim_{\varepsilon\to 0} T^2(f;\eta) = 0 \tag{27}$$

(again the limits may be even reversed); this completes the proof of the theorem. ■

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